Instability of frozen-in states in synchronous Hebbian neural networks

2008 J. Phys. A: Math. Theor. 41 265001

(http://iopscience.iop.org/1751-8121/41/26/265001)

J. Phys. A: Math. Theor. **41** (2008) 265001 (13pp) doi:10.1088/1751-8113/41/26/265001

Instability of frozen-in states in synchronous Hebbian neural networks

F L Metz and W K Theumann

Instituto de Física, Universidade Federal do Rio Grande do Sul, Caixa Postal 15051, 91501-970 Porto Alegre, Brazil

E-mail: fmetz@if.ufrgs.br and theumann@if.ufrgs.br

Received 4 April 2008, in final form 12 May 2008
Published 9 June 2008
Online at stacks.iop.org/JPhysA/41/265001

Abstract

The full dynamics of a synchronous recurrent neural network model with Ising binary units and a Hebbian learning rule with a finite self-interaction is studied in order to determine the stability to synaptic and stochastic noise of frozen-in states that appear in the absence of both kinds of noise. Both the numerical simulation procedure of Eissfeller and Opper and a new alternative procedure that allows us to follow the dynamics over larger time scales have been used in this work. It is shown that synaptic noise destabilizes the frozen-in states and yields either retrieval or paramagnetic states for not too large stochastic noise. The indications are that the same results may follow in the absence of synaptic noise, for low stochastic noise.

PACS numbers: 75.10.Hk, 87.18.Sn, 02.50.−r

1. Introduction

The dynamics of recurrent neural network models with synchronous updating has been of interest ever since the work of Little [1–3]. It is expected that Little's model with binary units and symmetric Hebbian couplings should exhibit either cycles of period two or fixed-point attractors and it has been suggested that the cycles could arise in statistical mechanics from a duplication of phase space by means of two state variables for every unit. An equilibrium analysis, based on replica symmetry, predicted that period-two cycles associated with a full spin flip (all units changing sign at each time step) should occur in a zero-temperature paramagnetic phase. This was recognized as an unphysical phase apparently disconnected from the ordinary high temperature paramagnetic phase. A somewhat large negative self-interaction J_0 (the diagonal elements of the synaptic matrix) between units turned out to be necessary for this cyclic solution with full spin flip to appear [2].

Early numerical simulations failed to show the presence of those cycles and led to the conjecture that the problem with the statistical mechanics approach could be due to the assumption of replica symmetry. Indeed, the entropy of this phase was found to be negative, going to $-\infty$ as $T \to 0$ [2], a feature that usually goes together with replica symmetry [4]. In a further work, a zero-temperature calculation of the average number of cycles as a function of the fraction of flipping spins demonstrated that period-two cycles with full spin flip are by far the most common type of cycles below a storage ratio of patterns $\alpha \approx 0.7$ [5]. However, the usual calculation of the average number of metastable states does not say anything about the macroscopic properties, in particular about the overlap with a chosen pattern.

Fixed-point solutions for the macroscopic parameters that emerge from a dynamics with a finite fraction of spins changing sign at each time step are common, but not exclusive, to the phase diagram of Little's model. Indeed, they also appear in recent studies of the three-state Ising and Blume–Emery–Griffiths neural network models with synchronous updating. These models can be thought as extensions of Little's model with generalized synaptic interactions and multi-state units [6]. Only a small fraction of neurons that change sign appear to be involved in the stationary states [7, 8], and the work in those studies was mainly devoted to fixed-point solutions for the macroscopic parameters. On the other hand, stationary period-two solutions for the macroscopic parameters, that arise from a fraction of units changing sign at each time step, have been found in the recent work on the synchronous dynamics of symmetric sequence processing without a self-interaction [9]. The relation between the dynamics of the macroscopic parameters and the fraction of units that change sign at each time step is crucial to understand the behavior of synchronous networks.

A dynamical study may be helpful to gain further insight into the asymptotic behavior of Little's model. So far, only the results of an approximate synchronous dynamics are available, which become exact for an asymmetrically random diluted network in the limit of extreme dilution [10]. This dynamics predicts that, at $T = 0$ and $\alpha = 0$, either frozen-in fixed points or frozen-in cycles of period two may appear when $|J_0| > |m_0|$, for a positive or a negative self-interaction, respectively, where m_0 is the initial overlap with a chosen pattern. On the other hand, there is a flow to a fixed point $m_t = 1$ reached in time t, when $|J_0| < |m_0|$. Thus, both the size and the sign of the self-interaction play a crucial role and the same dynamics, for small nonzero α, predicts a flow either to a paramagnetic or to a retrieval state.

Frozen-in states do not evolve in time and they are an undesirable feature for associative memory. In the present context, they are states in which the overlap either remains fixed at m_0 (frozen-in fixed point) or oscillates between m_0 and $-m_0$ (frozen-in cycle of period two). The correlation function between two consecutive time steps, related to the fraction of flipping spins, is $C_{t,t-1} = 1$ for a frozen-in fixed point and $C_{t,t-1} = -1$ for a frozen-in cycle of period two. The question that arises is if the frozen-in states actually are stable in Little's fully connected model subject to synaptic and/or stochastic noise, and the main purpose of this paper is to deal with that issue following the transient behavior of the dynamics that yields the stationary states of the network. The full dynamics of the Hebbian synchronous recurrent network of binary units, either at zero or at finite temperature with stochastic noise, including eventually the region where period-two cycles dominate has, apparently, not been carried out before. We make use of a generating functional approach (GFA) [11, 12], exact in the mean-field limit of an infinitely large system, combined with the numerical simulation procedure of Eissfeller and Opper (EO), based on the GFA for the dynamics of disordered spin systems. This is a procedure free of finite size effects [13, 14].

The problem with spin-glass-like models (neural networks among them) is that the dynamics may be very slow. Due to that, in particular for not too small $|J_0|$ in both cyclic and in retrieval states, the numerical simulation may require a large number of time steps in order to reach a stationary behavior, making the computation prohibitive with the EO procedure. A further purpose of the paper is to overcome this problem by means of an alternative procedure that is introduced in this work and which consists mainly of two features. One is the neglect of memory effects that involve the response function in the self-energy term of the single-site effective field, which amounts to a signal-to-noise approximation (SNA) [15]. The second feature consists in doing analytically the summation over states in the GFA. Thereby one accounts for the microscopic variables in the calculation of the dynamics of the macroscopic properties in a way that involves a much smaller number of variables. The procedure is exact for $\alpha = 0$, and yields an approximation for small α, that can be done in a short computing time up to a large number of time steps producing results in an excellent agreement with the procedure of EO for finite T and sufficiently large values of $|J_0|$. We make use of both our and the EO procedure in what follows.

The outline of the paper is the following. In section 2, we introduce the model and present a brief summary of the now well known GFA supplemented with an adaptation of the EO procedure to our system. In section 3 we discuss our alternative procedure and in section 4 we show the results of the transient dynamics up to the asymptotic states for the overlap and the correlation function between two consecutive times. We conclude with a summary and a further discussion in section 5.

2. The model and the generating functional approach

We consider a network of N Ising neurons in a microscopic state $\boldsymbol{\sigma}(t) = \{\sigma_1(t), \ldots, \sigma_N(t)\}$, at the time step t in which each $\sigma_i(t) = \pm 1$ represents an active or inactive neuron, respectively. The states of all neurons are updated simultaneously at each discrete time step according to the alignment of each spin with its local field

$$h_i(t) = \sum_j J_{ij}\sigma_j(t) + \theta_i(t), \tag{1}$$

following a microscopic stochastic single spin-flip dynamics with transition probability

$$w[\sigma_i(t+1)|h_i(t)] = \tfrac{1}{2}\{1 + \sigma_i(t+1)\tanh[\beta h_i(t)]\}, \tag{2}$$

ruled by the noise-control parameter $\beta = T^{-1}$, where T is the synaptic noise. The dynamics is a deterministic one when $T = 0$ and fully random when $T = \infty$. In the former case, $\sigma_i(t+1) = \mathrm{sgn}[h_i(t)]$. Here, $\theta_i(t)$ is an external stimulus and J_{ij} is the synaptic coupling

$$\begin{aligned} J_{ij} &= \frac{1}{N}\sum_{\mu=1}^{p}\xi_i^{\mu}\xi_j^{\mu}, \qquad i \neq j, \\ &= J_0, \qquad i = j, \end{aligned} \tag{3}$$

between units i and j, which is assumed to have a Hebbian form when $i \neq j$, with a macroscopic set $\boldsymbol{\xi}^{\mu} = \left(\xi_1^{\mu}, \ldots, \xi_N^{\mu}\right)$, $\mu = 1, \ldots, p$ of $p = \alpha N$ independent and identically distributed quenched random patterns. Each $\xi_i^{\mu} = \pm 1$ with probability $\frac{1}{2}$ and J_0 is a variable self-interaction. The latter plays a crucial role leading either to fixed-point or cyclic behavior.

The dynamical evolution of the system is described by the moment generating functional [12]

$$\begin{aligned} Z(\psi) &= \left\langle \exp\left[-\mathrm{i}\sum_i\sum_{s=0}^{t}\psi_i(s)\sigma_i(s)\right]\right\rangle \\ &= \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t)} \mathrm{Prob}[\boldsymbol{\sigma}(0), \ldots, \boldsymbol{\sigma}(t)]\exp\left[-\mathrm{i}\sum_i\sum_{s=0}^{t}\psi_i(s)\sigma_i(s)\right], \end{aligned} \tag{4}$$

for a finite number of time steps t, where $\psi(t) = (\psi_1(t), \ldots, \psi_N(t))$ is a set of auxiliary variables that generates averages of moments of the states and the brackets denote an average over all possible paths of states with probability

$$\mathrm{Prob}[\boldsymbol{\sigma}(0), \ldots, \boldsymbol{\sigma}(t)] = p[\boldsymbol{\sigma}(0)] \prod_{s=0}^{t-1} \prod_i \frac{\exp[\beta\sigma_i(s+1)h_i(s)]}{2\cosh[\beta h_i(s)]} \tag{5}$$

that follows from (2). Assuming that for $N \to \infty$ only the statistical properties of the stored patterns will influence the macroscopic behavior, one obtains the relevant quantities which are the overlap $m_1(t)$ of O(1) with the condensed pattern ξ^1, say, the two-time correlation function $C(t, t')$ and the response function $G(t, t')$, given by

$$m_1(t) = \frac{1}{N}\sum_i \overline{\xi_i^1\langle\sigma_i(t)\rangle} = \lim_{\psi\to 0} \frac{\mathrm{i}}{N}\sum_i \xi_i^1 \frac{\partial\overline{Z(\psi)}}{\partial\psi_i(t)}, \tag{6}$$

$$C(t, t') = \frac{1}{N}\sum_i \overline{\langle\sigma_i(t)\sigma_i(t')\rangle} = -\lim_{\psi\to 0} \frac{1}{N}\sum_i \frac{\partial^2\overline{Z(\psi)}}{\partial\psi_i(t')\psi_i(t)}, \tag{7}$$

and

$$G(t, t') = \frac{1}{N}\sum_i \frac{\partial\overline{\langle\sigma_i(t)\rangle}}{\partial\theta_i(t')} = \mathrm{i}\lim_{\psi\to 0} \frac{1}{N}\sum_i \frac{\partial^2\overline{Z(\psi)}}{\partial\theta_i(t')\psi_i(t)}(t' < t), \tag{8}$$

where the bar denotes the configurational average with the non-condensed patters $\boldsymbol{\xi}^\mu$, $\mu > 1$, and the restriction $t' < t$ is due to causality. Calling $q_0(t) = C(t, t-1)$, the fraction of flipping spins between two consecutive times becomes $[1 - q_0(t)]/2$.

Following now the standard procedure in which the disorder average is done before the sum over the neuron states one obtains exactly, in the large N limit, the generating functional [12, 13]

$$Z(\psi) = \Bigg\langle \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t)} p[\boldsymbol{\sigma}(0)] \exp\left[-\mathrm{i}\sum_i\sum_{s=0}^{t}\psi_i(s)\sigma_i(s)\right] \times \prod_i \prod_{s<t} \left\{\int \mathrm{d}h_i(s)\delta\big[h_i(s) - h_i^{\mathrm{eff}}(s)\big] w[\sigma_i(s+1)|h_i(s)]\right\}\Bigg\rangle_{\{\phi_i(s)\}} \tag{9}$$

in which $p[\boldsymbol{\sigma}(0)] = \prod_i p[\sigma_i(0)]$ is the probability of the initial microscopic configuration while $\langle\cdots\rangle_{\{\phi_i(t)\}}$ denotes an average over the correlated Gaussian random variables $\{\phi_i(t)\}$, with zero average and a correlation matrix given below. These random variables turn out to be uncorrelated on different sites and one is left with a single-site effective theory in which a neuron evolves in time according to the probability

$$w[\sigma(t+1)|h^{\mathrm{eff}}(t)] = \tfrac{1}{2}\{1 + \sigma(t+1)\tanh[\beta h^{\mathrm{eff}}(t)]\}, \tag{10}$$

with an effective local field given by

$$h^{\mathrm{eff}}(t) = \xi m(t) + J_0\sigma(t) + \alpha\sum_{t'<t} R(t, t')\sigma(t') + \sqrt{\alpha}\phi(t), \tag{11}$$

where we dropped the label of the condensed pattern and assumed that $\theta(t) = 0$. The two non-trivial contributions to the effective local field for $\alpha \neq 0$ come from a retarded self-interaction involving the matrix elements

$$R(t, t') = [\boldsymbol{G}(\boldsymbol{I} - \boldsymbol{G})^{-1}]_{t,t'} \tag{12}$$

and the zero average temporarily correlated Gaussian noise $\phi(t)$ with variance

$$S(t,t') = \langle \phi(t)\phi(t')\rangle_{\{\phi(t)\}} = [(\boldsymbol{I}-\boldsymbol{G})^{-1}\boldsymbol{C}(\boldsymbol{I}-\boldsymbol{G}^{\dagger})^{-1}]_{t,t'}. \tag{13}$$

Both these terms account for memory effects in the network. The generating functional (9) is the functional of Eissfeller and Opper that applies to Little's model and the specific algorithm that implements the numerical simulations is described in the literature [13, 14].

The dynamics of each of the macroscopic quantities, given by (6)–(8), is obtained from the statistics of the effective single neuron process through the average

$$\langle f(\boldsymbol{\sigma})\rangle = \int \mathrm{d}\phi\, \mathrm{P}(\phi) \sum_{\boldsymbol{\sigma}} \mathrm{P}(\boldsymbol{\sigma}|\phi) f(\boldsymbol{\sigma}), \tag{14}$$

where $\boldsymbol{\sigma} = \{\sigma(t)\}$ and $\phi = \{\phi(t)\}$ are now single-site vectors that follow a path in discrete times, and

$$\mathrm{P}(\boldsymbol{\sigma}|\phi) = p[\sigma(0)] \prod_{s<t} w[\sigma(s+1)|h^{\mathrm{eff}}(s)] \tag{15}$$

is the single-spin path probability given the Gaussian noise ϕ in the effective field, with a distribution

$$\mathrm{P}(\phi) = \frac{1}{\sqrt{(2\pi)^t \det \boldsymbol{S}}} \exp\left(-\tfrac{1}{2}\phi \cdot \boldsymbol{S}^{-1}\phi\right). \tag{16}$$

The macroscopic parameters (overlap, correlation and response function) can, in principle, be calculated in a closed form for any time step t but that becomes non-practical since it requires an increasingly large number of macroscopic quantities. Thus, analytic calculations are only feasible for the first few time steps and, as it turns out, also for the asymptotic stationary state, albeit under some conditions which are not always fulfilled.

In order to obtain the full dynamic description of the transients, we make use of the procedure of Eissfeller and Opper in which the effective single-site dynamics given by (10)–(13) is simulated by a Monte Carlo method. There are no finite-size effects, but a large number N_T of stochastic trajectories has to be generated for the single-site process and the macroscopic parameters can be obtained from the average

$$\langle f(\boldsymbol{\sigma})\rangle = \frac{1}{N_T}\sum_{a=1}^{N_T} f(\boldsymbol{\sigma}_a), \tag{17}$$

where $\boldsymbol{\sigma}_a$ denotes the set of spins along the path a. The number of stochastic trajectories N_T should not be confused with the number of neurons N, which goes to infinity. To keep the numerical errors small, a sufficiently large N_T must be used.

3. Alternative procedure

The alternative procedure consists in doing analytically the sum over the microscopic paths (states in discrete times) $\boldsymbol{\sigma}$, to start with, in order to have a much smaller set of parameters to iterate in the numerical calculations. Due to the presence of memory effects in the terms involving the retarded self-interaction $R(t,t')$, given by (12), which relate the state of the system at time t to that at all previous times, the full elimination of the microscopic parameters becomes a formidable task. In the following we consider the simplest approximation that consists in assuming that $\boldsymbol{G} = 0$, which is the so-called signal-to-noise approximation [15]. This implies, in turn, that the variance of the Gaussian correlated noise becomes $\boldsymbol{S} = \boldsymbol{C}$ according to (13).

With that approximation, the effective local field with the time step denoted as a subindex,

$$h_t = \xi m_t + J_0\sigma_t + \sqrt{\alpha}\phi_t \tag{18}$$

has still a stochastic Gaussian noise for nonzero α and a dependence on the spin variable σ_t at the same time. We assume an initial distribution $p(\sigma_0) = \frac{1}{2}(1+\sigma_0\xi m_0)$ and proceed with the summation over states in (14), which we denote as

$$g(\phi) = \sum_{\boldsymbol{\sigma}} \mathrm{P}(\boldsymbol{\sigma}|\phi) f(\boldsymbol{\sigma}). \tag{19}$$

The summation can be done by inserting the following integral representation:

$$\exp(\beta\sigma_{s+1}h_s) = \int_0^{2\pi} \frac{\mathrm{d}x_s}{\pi} \exp(\mathrm{i}x_s\sigma_{s+1}) \cosh(\beta h_s - \mathrm{i}x_s) \tag{20}$$

into (15), with the effective local field given by (18), in order to separate the dependence on the states at two consecutive times. The result, substituted in (19), yields

$$g(\phi) = \int_0^{2\pi} \left[\prod_{s=0}^{t-1} \frac{\mathrm{d}x_s}{2\pi}\right] \sum_{\sigma_0\ldots\sigma_t} p(\sigma_0) f(\boldsymbol{\sigma}) \exp\left(\mathrm{i}\sum_{s=0}^{t-1} x_s\sigma_{s+1}\right)$$
$$\times \prod_{s=0}^{t-1} [\cos x_s - \mathrm{i}\tanh(\beta h_s)\sin x_s]. \tag{21}$$

Since the argument of the exponential is linear with respect to the set $\{\sigma_s\}$, we can now do the sum over the states separately for each time in order to obtain the form of the function $g(\phi)$ in the case of the overlap and the two-time correlation function, with $f(\boldsymbol{\sigma})$ given by $f(\boldsymbol{\sigma}) = \xi\sigma_t$ and $f(\boldsymbol{\sigma}) = \sigma_t\sigma_s$ $(s = 0, \ldots, t-1)$, respectively. The drawback is the appearance of t multiple integrals over the full set $\{x_s\}$, which can ultimately be reduced to a pair of integrals over x_0 and x_{t-1} involving elements of either of the matrices

$$\boldsymbol{U} = \prod_{s=1}^{t-1} \boldsymbol{A}_s, \qquad \boldsymbol{V} = \left(\prod_{s=1}^{t'-1} \boldsymbol{A}_s\right) \boldsymbol{B}_{t'} \left(\prod_{s=t'+1}^{t-1} \boldsymbol{A}_s\right), \tag{22}$$

in the representation

$$|x_s\rangle = \begin{pmatrix} \cos x_s \\ \sin x_s \end{pmatrix}, \tag{23}$$

where

$$\boldsymbol{A}_s = \begin{pmatrix} 1 & -\frac{\mathrm{i}}{2}[\tanh\beta(\xi m_s + J_0 + \sqrt{\alpha}\phi_s) + \tanh\beta(\xi m_s - J_0 + \sqrt{\alpha}\phi_s)] \\ 0 & \frac{1}{2}[\tanh\beta(\xi m_s + J_0 + \sqrt{\alpha}\phi_s) - \tanh\beta(\xi m_s - J_0 + \sqrt{\alpha}\phi_s)] \end{pmatrix}, \tag{24}$$

$$\boldsymbol{B}_s = \begin{pmatrix} 0 & -\frac{\mathrm{i}}{2}[\tanh\beta(\xi m_s + J_0 + \sqrt{\alpha}\phi_s) - \tanh\beta(\xi m_s - J_0 + \sqrt{\alpha}\phi_s)] \\ \mathrm{i} & \frac{1}{2}[\tanh\beta(\xi m_s + J_0 + \sqrt{\alpha}\phi_s) + \tanh\beta(\xi m_s - J_0 + \sqrt{\alpha}\phi_s)] \end{pmatrix}. \tag{25}$$

Performing the discrete average over the pattern ξ, we obtain the explicit expressions for the macroscopic parameters, that is, for the overlap

$$m_t = \big\langle\big[\tfrac{1}{2}(1+m_0)\tanh\beta(m_0 + J_0 + \sqrt{\alpha}\phi_0) + \tfrac{1}{2}(1-m_0)\tanh\beta(m_0 - J_0$$
$$+ \sqrt{\alpha}\phi_0)\big]U_{22} + \mathrm{i}U_{12}\big\rangle_\phi, \tag{26}$$

for the correlation with the initial state

$$C_{t0} = \big\langle\big[\tfrac{1}{2}(1+m_0)\tanh\beta(m_0 + J_0 + \sqrt{\alpha}\phi_0)$$
$$- \tfrac{1}{2}(1-m_0)\tanh\beta(m_0 - J_0 + \sqrt{\alpha}\phi_0)\big]U_{22} + \mathrm{i}m_0U_{12}\big\rangle_\phi, \tag{27}$$

and for the correlation with the states at other times $0 < t' < t$,

$$C_{tt'} = \Big\langle \Big[\tfrac{1}{2}(1+m_0)\tanh\beta(m_0+J_0+\sqrt{\alpha}\phi_0) + \tfrac{1}{2}(1-m_0)\tanh\beta(m_0-J_0+\sqrt{\alpha}\phi_0)\Big] V_{22} + \mathrm{i}V_{12}\Big\rangle_\phi, \tag{28}$$

where $U_{i,j}$ and $V_{i,j}$ are the elements of the matrices $\boldsymbol{U}$ and $\boldsymbol{V}$. Finally, for the binary spins in this work, the diagonal elements are $C_{t,t} = 1$. Here, m_0 and ϕ_0 are the initial overlap and Gaussian noise, respectively, and the full correlation function is needed to calculate the average

$$\langle g(\phi)\rangle_\phi = \int \mathrm{d}\phi \frac{\exp\left(-\frac{1}{2}\phi\cdot \boldsymbol{C}^{-1}\phi\right)}{(2\pi)^{\frac{t}{2}}\sqrt{\det \boldsymbol{C}}} g(\phi). \tag{29}$$

Note that (26)–(28) are a set of coupled equations when $\alpha \neq 0$ involving the matrix $\boldsymbol{C}$ and the overlap at previous time steps through the matrices $\boldsymbol{U}$ and $\boldsymbol{V}$. The remaining Gaussian averages in the above equations have to be performed numerically replacing the integral over ϕ by a discrete average over a large number M of randomly generated functions of the set of correlated Gaussian variables $\{\phi_t^\lambda\}$, $\lambda = 1,\dots,M$, at each time step t, with mean zero and variance given by the correlation matrix $\boldsymbol{C}$.

Before presenting our numerical results we discuss briefly the case where $\alpha = 0$. Thus, in the absence of stochastic noise, the alternative procedure becomes exact and the average over ϕ drops out in (26)–(28) leading to the following two recursion relations,

$$m_{t+1} = \tfrac{1}{2}(1+m_t)\tanh\beta(m_t+J_0) + \tfrac{1}{2}(1-m_t)\tanh\beta(m_t-J_0) \tag{30}$$

and

$$C_{t+1,s} = \tfrac{1}{2}(C_{t,s}+m_s)\tanh\beta(m_t+J_0) - \tfrac{1}{2}(C_{t,s}-m_s)\tanh\beta(m_t-J_0), \tag{31}$$

for $s < t+1$. Setting $s = t$ and noting that $C_{t,t} = 1$, we get the two-consecutive time correlation function that yields the spin-flip order parameter $q_0(t) = C_{t+1,t}$ satisfying the equation

$$q_0(t) = \tfrac{1}{2}(1+m_t)\tanh\beta(m_t+J_0) - \tfrac{1}{2}(1-m_t)\tanh\beta(m_t-J_0). \tag{32}$$

Thus, (30) and (32) are extensions for all T of the equations obtained before for $T = 0$ [10]. The asymptotic stationary order parameters are given by the fixed-point solutions of these equations, which become [2]

$$m = \frac{\sinh(2\beta m)}{\cosh(2\beta m)+\exp(-2\beta J_0)} \tag{33}$$

and

$$q_0 = \frac{\cosh(2\beta m)-\exp(-2\beta J_0)}{\cosh(2\beta m)+\exp(-2\beta J_0)}. \tag{34}$$

In the $T = 0$ limit,

$$\begin{aligned} |J_0| > |m_0| &\to m_t = m_0, && q_0 = 1 && \text{if } J_0 > 0 \\ &\to m_t = (-1)^t m_0, && q_0 = -1 && \text{if } J_0 < 0 \\ |J_0| < |m_0| &\to m_t = \mathrm{sgn}(m_0), && q_0 = 1. \end{aligned} \tag{35}$$

Thus, at $T = 0$ and $\alpha = 0$, the system appears in a frozen-in state either in the initial overlap when $J_0 > 0$ or in a cycle when $J_0 < 0$, in the first case, or else it flows to the retrieval state in the second case. It is in the first case that we are interested in what follows. Clearly, in that case the system is no longer useful as an associative memory but it could again become useful in the presence of synaptic and/or stochastic noise and it would be interesting to see how this occurs and we discuss that next. When the dynamics is not too slow one may employ the procedure of EO, but otherwise we have to resort to our alternative approximate dynamics.

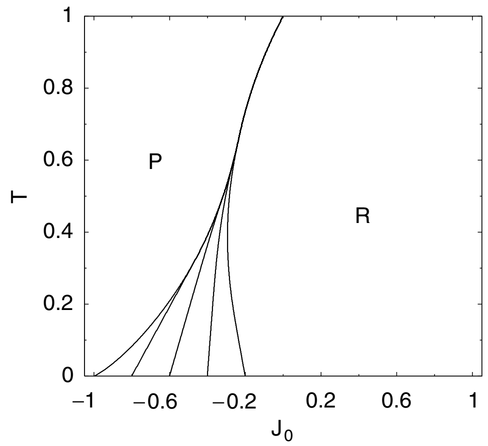


Figure 1. Phase diagram for $\alpha = 0$ with initial overlaps $m_0 = 1, 0.8, 0.6, 0.4$ and 0.2, from left to right.

4. Results

We start with the results for $\alpha = 0$ and any T, for which the alternative procedure is exact. In figure 1 we show the (J_0, T) phase diagram of stationary states for various initial overlaps, as indicated, obtained by means of the iteration of (30). We also analyzed the stationary states of the correlation function between two consecutive times, by means of (32). There is a paramagnetic phase (P) to the left of the curves with $m = 0$ and $q_0 > -1$ for all T, except at $T = 0$, where there is a state of frozen-in cycles with overlap $m_t = (-1)^t m_0$ and $q_0 = -1$. The network evolves to a retrieval fixed point within the phase R for all $T > 0$, with an asymptotic overlap $m \simeq 1$ for low T and $q_0 < 1$. For $T = 0$, there is either a state of frozen-in fixed points in phase R with $m = m_0$ and $q_0 = 1$, if $J_0 > m_0$, or a retrieval state with $m = 1$ and $q_0 = 1$, if $J_0 < m_0$, in accordance with (35). The phase boundary obtained in the case of the initial overlap $m_0 = 0.5$ is precisely the same as that obtained from the equality of the free energies in the equilibrium analysis, where the appearance of a tricritical point and other features have been discussed [2].

In order to illustrate the instability to synaptic noise of the frozen-in fixed point in phase R, when $\alpha = 0$, and the interesting transient crossover to retrieval behavior, we show in figure 2 the dynamical evolution of the overlap and of the two-time correlation function for $J_0 = 0.8$, $T = 0.08$ and initial overlap $m_0 = 0.4$. The results were obtained by means of the alternative procedure and, for comparison, we also show the results with the EO procedure. For these values of the parameters, well within the domain where $J_0 > m_0$ already for a moderate value of J_0, a large number of time steps is needed in order to reach the asymptotic behavior and even longer times are necessary for larger values of J_0 and/or smaller values of T. In contrast, the EO procedure yields inconclusive results within reasonable computing time, that could lead to wrong conclusions. With the exception of the dip around $t \simeq 1575$, which is an indication of crossover behavior in the overlap, the consecutive-time correlation function $C_{t,t-1}$ is very close to but not exactly equal to one through all the dynamics, which indicates that there is almost always a very small fraction of flipping spins, except at the crossover where that fraction is about 5% over a few number of steps.

We consider now the instability to synaptic noise of the frozen-in cycles of period two in phase P, when $\alpha = 0$. The results obtained with the alternative approach and, for comparison

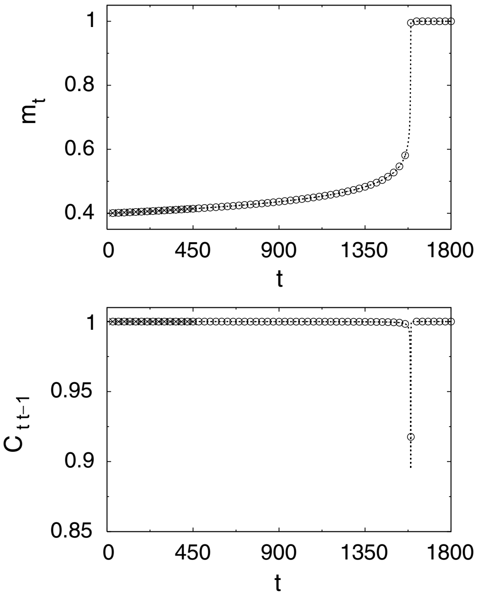


Figure 2. Overlap and two-time correlation function for $T = 0.08$, $J_0 = 0.8$, $m_0 = 0.4$ and $\alpha = 0$. Superimposed are the results with the EO procedure (dark circles).

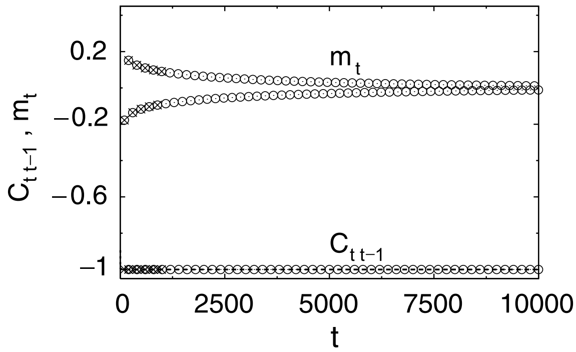
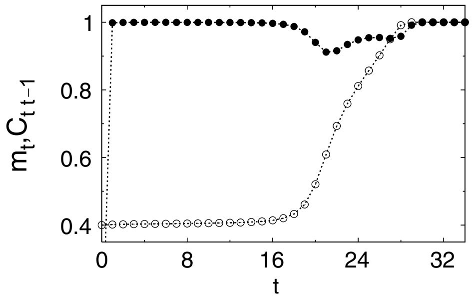
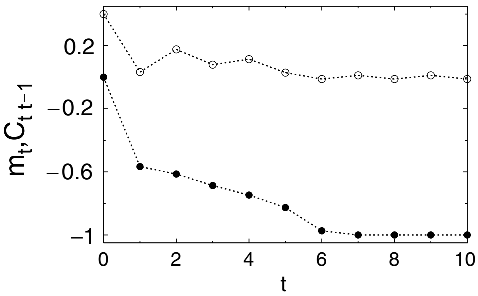


Figure 3. Overlap and two-time correlation function for $T = 0.08$, $J_0 = -0.5$, $m_0 = 0.4$ and $\alpha = 0$. Superimposed are the results with the EO procedure (dark circles).

with the EO procedure, are shown in figure 3 for $J_0 = -0.5$, $T = 0.08$ and initial overlap $m_0 = 0.4$. The oscillating overlap decreases continuously to zero with increasing t, now with the parameters in the domain where $|J_0| > m_0$ and $J_0 < 0$ and, again, a large number of time steps is needed in order to reach a vanishing overlap characteristic of a paramagnetic phase. The two-time correlation function $C_{t,t-1}$ is very slightly larger than -1 through all the dynamics, which indicates the permanent presence of a small fraction of units that remains frozen between two consecutive times, causing a continuous decrease in the amplitude of the cycles as time evolves. It turns out that the decrease is slower for larger values of $|J_0|$ and/or smaller values of T. We remark, again, that the convergence to the asymptotic behavior is rather slow and that results based on a much shorter time scale could lead to the wrong conclusion that there are cycles for finite T.



(*a*) $\alpha = 0.005$ and $J_0 = 0.6$.



(*b*) $\alpha = 0.04$ and $J_0 = -0.5$.

Figure 4. Overlap (open circles) and two-time correlation function (full circles) at $T = 0$, initial overlap $m_0 = 0.4$ and $N_T = 5 \times 10^5$ stochastic trajectories. The lines are a guide to the eye.

Consider next the situation for $|J_0| > |m_0|$ in the absence of synaptic noise ($T = 0$) and $\alpha \neq 0$. In this case we run into difficulties with the EO procedure due to the vanishing denominator in (16) after a relatively small number of time steps, and the alternative dynamics turned out to be not a good approximation in this situation. Nevertheless, one can still draw conclusions from the calculations for a small number of time steps using the EO procedure, as we show next.

In figure 4(*a*) we illustrate the evolution of the overlap and of the correlation function $C_{t,t-1}$ for $J_0 = 0.6$, $\alpha = 0.005$ and $m_0 = 0.4$, within the domain where $J_0 > m_0$. There is a transient crossover from a frozen-in to a retrieval state which takes place by means of an increase in the fraction of flipping spins, shown by the dip in $C_{t,t-1}$, similar to the behavior discussed above. The correlation function between consecutive time steps is very close to but smaller than one through all the dynamics, indicating a very small but finite fraction of flipping spins outside the crossover region. It takes a longer time interval to reach the crossover the bigger the value of J_0 and/or the smaller the value of α. For values of J_0 closer to m_0 the transient crossover takes place for even smaller values of α, which suggests that this behavior should occur for any α. This, and the result obtained above, illustrates the interesting feature of the emergence of retrieval behavior in the presence of a small amount of either synaptic or stochastic noise, just enough to draw the network from the frozen-in state.

For a larger $\alpha \simeq 0.3$ and the other parameters remaining the same, we still find that $C_{t,t-1} < 1$ through all the dynamics, but after an increase over the first few time steps the overlap follows a very slow decrease as time evolves until a remanent value is attained, which is characteristic of spin-glass behavior [16].

In figure 4(*b*) we exhibit the dynamical behavior for $J_0 = -0.5$, $\alpha = 0.04$ and $m_0 = 0.4$, in which the amplitude of the oscillating overlap decreases rapidly and goes to zero. At the same time, the correlation function $C_{t,t-1}$ decays to a value very close to -1 indicating an almost full spin flip already after 7 time steps. For smaller values of $|J_0|$ and α, within the domain where $|J_0| > m_0$, a similar behavior is reached within a shorter number of time steps. This behavior is reminiscent of the paramagnetic phase in the presence of synaptic noise discussed above when $\alpha = 0$. A stationary state with vanishing overlap for $T = 0$, $J_0 < 0$ and finite α is in qualitative agreement with a paramagnetic phase found in the equilibrium replica symmetric approach [2]. It is worth noting that a paramagnetic state has also been found in an asymmetric extremely diluted synchronous network in the presence of stochastic noise at $T = 0$ [10] and $J_0 < 0$. We remind the reader, however, that we are dealing here with a fully connected network with symmetric interactions.

In contrast, for a slightly larger value of α, and the other parameters being the same, $C_{t,t-1}$ no longer becomes -1 and, instead, evolves toward a positive stationary value, indicating a partial spin flip, together with an oscillating overlap that decreases slowly until a finite remanent value is reached, characteristic of spin-glass behavior.

Since our results suggest that either synaptic or stochastic noise, with finite T or α respectively, play a similar role in the dynamics when $|J_0| > |m_0|$ in drawing the network from the frozen-in states, one can expect that this will continue to be the case in the presence of both kinds of noise. We checked this explicitly for $m_0 = 0.4$, $T = 0.08$, $\alpha = 0.003$ and two values of J_0. As expected, for $J_0 = 0.8$ we get the transient crossover from a frozen-in state to retrieval behavior and, for $J_0 = -0.5$, we find a continuous decreasing amplitude of the oscillating overlap as the time evolves, until a vanishing overlap is reached. In both cases we obtain $|C_{t,t-1}| < 1$ through the dynamics, indicating either a small but finite fraction of flipping units or an almost full spin flip, for $C_{t,t-1} > 0$ or $C_{t,t-1} < 0$, respectively. Due to the small values of T and α, it is necessary to go to larger times than those where the EO procedure is practically applicable, and we resorted for this purpose to the alternative approach which should be a good approximation for the value of α used. In fact, for the smaller times, there is very good agreement between the EO procedure with $N_T = 5 \times 10^5$ and the alternative approach with $M = 100$.

5. Summary and conclusions

We used the numerical simulation procedure of Eissfeller and Opper and an alternative procedure developed in this work, both based on a generating functional approach, in order to study the effects of synaptic and stochastic noise on the dynamical evolution of Little's model of a synchronous neural network with binary neurons and a finite self-interaction J_0. We analyzed the stability to both kinds of noise of frozen-in fixed points and frozen-in cycles that occupy a large part of the space of parameters and that were found in a previous work [10]. This is a crucial issue in the range of parameters when there is a sizeable self-interaction such that $|J_0| > |m_0|$, for which there is no guaranty that there will be a flow to a retrieval fixed point, even if $J_0 > 0$. That flow may be expected only if $|J_0| < |m_0|$.

We found that the frozen-in states that appear at $T = 0$ and $\alpha = 0$ are unstable to synaptic noise, with or without stochastic noise, leading either to retrieval or to paramagnetic states, for $J_0 > 0$ or $J_0 < 0$, respectively. Our work also suggests the instability of the frozen-in states to stochastic noise in the absence of synaptic noise, with the same kind of final states. This implies the absence of period-two cycles in any finite region of the phase diagram with T and/or α different from zero.

One has to be certain that the true asymptotic state is reached in the course of the dynamics. It is in order to deal with very long transients that already appear for moderate values of $|J_0|$ and low values of synaptic and/or stochastic noise, which are heavily time consuming in the computations with the EO procedure, that we developed our alternative dynamical procedure within the generating functional approach. The procedure consists in neglecting the retarded self-interaction term in the local field and performing explicitly the summation over states that enters into the calculation of the macroscopic quantities of interest, and it amounts to the numerical solution of a set of equations for the relevant macroscopic parameters that involves a Gaussian correlated average, and it becomes exact in the absence of stochastic noise. Thus, we can perform the computations over a much longer number of time steps in a shorter computing time than the procedure of Eissfeller and Opper. The procedure also yields results in very good agreement with the EO procedure in the case of small stochastic noise, finite T and large values of $|J_0|$, within the time scale where both procedures are applicable.

Finally, some concern about our alternative procedure. Although it is exact only in the absence of stochastic noise, that is for $\alpha = 0$, our results for $T > 0$ and a rather small $\alpha \neq 0$ should be reliable and it does not seem to be justified at this stage, where we are not concerned with full phase diagrams, to include memory effects in the retarded self-interaction due to the states of the units at other than just the previous time step. On the other hand, reasonable calculations can be done with the alternative procedure over longer time scales which would be needed to obtain results for larger ratios of $|J_0|/m_0$.

Despite the fact that the use of our alternative dynamics is justified for sufficiently small stochastic noise (small values of α), it may be interesting to consider its extension for a larger amount of noise. Even an approximate estimate of the retarded self-interaction $R(t, t')$, based on the dominating features of the response function $G(t, t')$, would be useful for that purpose. A hint in that direction could come from simulations of Eissfeller and Opper for $G(t, t')$. Preliminary calculations in that direction, for small nonzero values of T and α, show that the response function is close to zero for most times, except for the presence of two peaks.

It is worth pointing out that the alternative procedure could be useful in the dynamics of other synchronous neural network models with a finite self-interaction J_0. The procedure should also be useful for the study of the dynamics of other disordered systems. Furthermore, the procedure can also be applied to neural network models with $J_0 = 0$, including in the memory term of the effective local field (11) only the term which relates the state of the system at time t to that at time $t - 2$. In this case, it would be necessary to obtain an equation for the response function. This, and related issues will be explored in a separate work.

Acknowledgments

We thank J F Fontanari for an illuminating discussion. The work of one of the authors (WKT) was financially supported, in part, by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazil. A grant from FAPERGS (Fundação de Amparo à Pesquisa do Estado de Rio Grande do Sul), Brazil, to the same author is gratefully acknowledged. F L Metz acknowledges a fellowship from CNPq.

References

[1] Little W A 1974 *Math. Biosci.* **19** 101
Little W A and Shaw G L 1978 *Math. Biosci.* **39** 281
[2] Fontanari J F and Köberle R 1987 *Phys. Rev.* A **36** 2475
Fontanari J F and Köberle R 1988 *J. Phys. France* **49** 13
Fontanari J F and Köberle R 1988 *J. Phys. A: Math. Gen.* **21** L259

[3] Coolen A C C 2001 *Handbook of Biological Physics IV: Neuro-Informatics and Neural Modeling* ed F Moss and S Gielen (Amsterdam: Elsevier) p 553
[4] Mezard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[5] Fontanari J F 1997 *J. Phys. A: Math. Gen.* **30** 6655
[6] Bollé D 2004 *Advances in Condensed Matter and Statistical Mechanics* ed E Korutcheva and R Cuerno (New York: Nova Science Publishers) p 319
[7] Bollé D and Busquets Blanco J 2005 *Eur. Phys. J.* B **47** 281
[8] Bollé D, Erichsen R Jr and Verbeiren T 2006 *Physica* A **368** 311
[9] Metz F L and Theumann W K in preparation
[10] Fontanari J F 1988 *PhD Thesis* University of São Paulo, São Carlos, Brazil
[11] De Dominicis C 1978 *Phys. Rev.* B **18** 4913
[12] Coolen A C C 2001 *Handbook of Biological Physics IV: Neuro-Informatics and Neural Modeling* ed F Moss and S Gielen (Amsterdam: Elsevier) p 619
[13] Eissfeller H and Opper M 1992 *Phys. Rev. Lett.* **68** 2094
Eissfeller H and Opper M 1994 *Phys. Rev.* E **50** 709
[14] Verbeiren T 2003 *PhD thesis* K. U. Leuven, Belgium
[15] Bollé D, Busquets Blanco J and Verbeiren T 2004 *J. Phys. A: Math. Gen.* **37** 1951
[16] Scharnagel A, Opper M and Kinzel W 1995 *J. Phys. A: Math. Gen.* **28** 5721